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MHV vertices and scattering amplitudes in gauge theory

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Abstract: The generic googly amplitudes in gauge theory are computed by using the Cachazo-Svrcek-Witten approach to perturbative calculation in gauge theory and the results are in agreement with the previously well-known ones. Within this approach we also discuss the parity transformation, charge conjugation and the dual Ward identity. We also extend this calculation to include fermions and the googly amplitudes with a single quark-anti-quark pair are obtained correctly from fermionic MHV vertices. At the end we briefly discuss the possible extension of this approach to gravity.

Keywords: Nonperturbative Effects, Strong Coupling Expansion, QCD.

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Contents

1. Introduction

Recently Witten [1] found a deep connection between the perturbative gauge theory and string theory in twistor space [2]. Based on this work, Cachazo, Svrcek and Witten reformulated the perturbative calculation of the scattering amplitudes in Yang-Mills theory by using the off shell MHV vertices [3]. The MHV vertices they used are the usual tree level MHV scattering amplitudes in gauge theory [4, 5], continued off shell in a particular fashion as given in [3]. (For references on perturbative calculations, see for example [6]–[10]. The 2 dimensional origin of the MHV amplitudes in gauge theory was first given in [11].) Some sample calculations were done in [3], sometimes with the help of symbolic manipulation. The correctness of the rules was partially verified by reproducing the known results for small number of gluons [6].

In a previous work [12] (for recent works, see [13]–[29]), by following the new approach of [3], one of the authors computed the exceptionally simple amplitudes with two positive helicity gluons and an arbitrary number of negative helicity ones, called googly amplitudes in [3]. These amplitudes were calculated from the string theory in [13]. In the special case when the two positive helicity particles are adjacent, the result was shown to be in agreement with the well-known result [4]. In this paper we will calculate the generic googly amplitudes. By reproducing the previously well known results, our calculation gives a quite strong support to the Cachazo-Svrcek-Witten (CSW for short) proposal.

Although these calculations gave strong support to the CSW approach to the perturbative calculation in gauge theory, a direct proof for the equivalence to the usual Feynman rules seems hopeless. On the other hand, the twistor string theory approach [1] gives a different but rather compact expression for the tree level amplitude of gluons [13, 14]. It was proved [14] that this expression satisfies most of the requirements for the tree amplitude of gluons. In [22] it was argued that the string representation (a connected instanton) of the tree amplitude can indeed be decomposed into a summation of different MHV amplitudes (a set of disconnected instantons), giving precisely the CSW rules of perturbative calculations. Encouraged by the success of the googly amplitude calculation we will present some general discussions on the MHV diagrams in gauge theory. We note that in the calculation of the googly amplitudes by using the CSW rules, we only need the MHV vertices with 3 lines and the MHV vertices with 4 lines. So in a parity conserved theory the higher point (5 or more) MHV amplitudes can be obtained by parity transformation from the googly amplitudes and so the higher point MHV vertices can't be chosen arbitrarily. We also prove that the CSW rules satisfy the dual ward identity and the charge conjugation identity [6].

The CSW proposal [3] can also be extended to gauge theory with fermions (see also [21]). Although some of the MHV (and googly) fermionic amplitudes can be easily obtained by supersymmetric Ward identities [30, 31, 6] we think it is still worthy to compute these amplitudes directly because the general non-MHV (googly) ammplitudes cannot be determined in terms of amplitudes with gluons only and should be computed separately.[1] We will compute the googly amplitudes with fermions by extending the CSW rules with MHV vertices. For illustration we consider only the simplest case of a single quark-anti-quark pair. The general cases, including the supersymmetric case with gluinos, will be discussed in a separate publication [33]. Some generic non-MHV fermionic amplitudes were also computed in [21, 32].

Another interesting question is whether the CSW approach can be extended to theories with gravity. A naive extension of the the CSW rules to graviton doesn't seem to work [24]. Nevertheless we believe that some similar rules for graviton must exist, given the simplicity of the MHV graviton amplitude [34] and the KLT relations between gauge theory and gravity [35]. In [26], Berkovits and Witten put forward a similar connection between the superconformal supergravity and closed string theory in twistor space. This may also suggest an extension of the CSW rules. In the last section we will present our partial successful and un-successful attempts to this problem. In particular we give a simple rule for the calculation of the off-shell amplitude with a single positive helicity graviton. This amplitude is proportional to the square of the (only) off-shell momentum and it is vanishing on shell.

This paper is organized as follows. In section 2, we present the computation of the generic googly amplitudes. In section 3, we give some general discussions on the MHV diagrams. We define precisely how the parity transformation operates in the CSW approach. We also prove the charge conjugation identity and the dual Ward identity in this section. In

[1] In an early version of this paper we don't say this explicitly although we suspect this is the case. See the footnote of the recent paper [32]. We thank the referee to point out this to us.

section 4, we extend the CSW rules to include fermions and compute the googly amplitude with a single quark and anti-quark pair. In the last section we make some investigations on the graviton MHV diagrams. Some technical proofs are relegated to two appendices A and B.

2. The generic googly amplitudes in gauge theory

First let us recall the rules for calculating tree level gauge theory amplitudes as proposed in [3]. Here we follow the presentation given in [12] closely. We will use the convention that all momenta are outgoing. By MHV we always mean an amplitude with precisely two gluons of negative helicity. If the two gluons of negative helicity are labelled as r, s (which may be any integers from 1 to n), the MHV vertices (or amplitudes) are given as follows:

$$V_n = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \,. \tag{2.1}$$

For an on shell (massless) gluon, the momentum in bispinor basis is given as:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} p_\mu = \lambda_a \tilde{\lambda}_{\dot{a}} \,. \tag{2.2}$$

For an off shell momentum, we can no longer define λ_a as above. The off-shell continuation given in [3] is to choose an arbitrary spinor $\tilde{\eta}^{\dot{a}}$ and then to define λ_a as follows:

$$\lambda_a = p_{a\dot{a}} \tilde{\eta}^{\dot{a}} \,. \tag{2.3}$$

For an on shell momentum p, we will use the notation λ_{pa} which is proportional to λ_a:

$$\lambda_{pa} \equiv p_{a\dot{a}} \tilde{\eta}^{\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \tilde{\eta}^{\dot{a}} \equiv \lambda_a \phi_p \,. \tag{2.4}$$

As demonstrated in [3], it is consistent to use the same $\tilde{\eta}$ for all the off shell lines (or momenta). The final result is independent of $\tilde{\eta}$.

By using only MHV vertices, one can build a tree diagram by connecting MHV vertices with propagators. For the propagator of momentum p, we assign a factor $1/p^2$. Any possible diagram (involving only MHV vertices) will contribute to the amplitude. As proved in [3], a tree level amplitude with n_- external gluons of negative helicity must be obtained from an MHV tree diagram with $n_- - 1$ vertices. Another relation was given in [12],

$$n_+ = \sum_i n_i (i-3) + 1 \,, \tag{2.5}$$

when n_+ is the number of the external gluons with positive helicity, and n_i is the number of the vertices with exactly i lines. The other relation stated in the above is:

$$n_- = \sum_i n_i + 1 \,. \tag{2.6}$$

From eq. (2.5) we can see that a tree level amplitude with n_+ external gluons of positive helicity will have no contribution from any diagram containing an MHV vertex with more

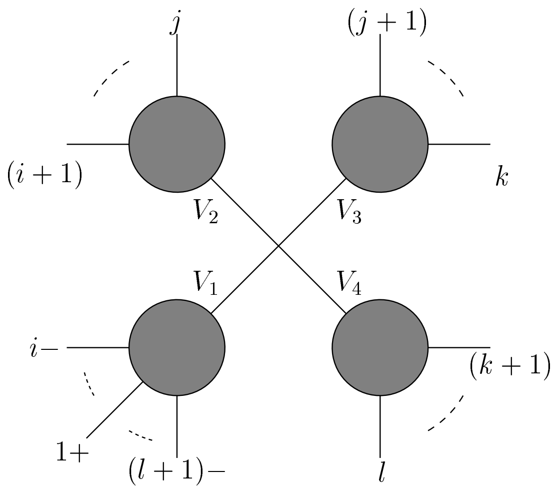

Figure 1: The diagram decomposition for the generic googly amplitude. We note that there is only one 4 gluon vertex.

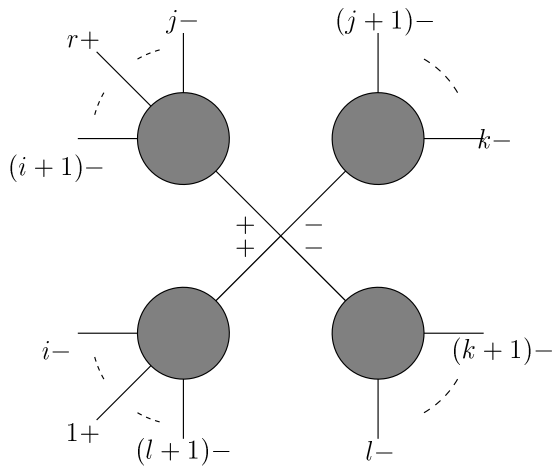

Figure 2: The diagram decomposition for the generic googly amplitude when $(i+1) \leq r \leq j$.

than $n_+ + 2$ lines (not necessarily all internal). For the googly amplitude we have $n_+ = 2$. Any contributing diagram will have exactly one MHV vertex with 4 lines. The rest MHV vertices are all with 3 lines.

In order to compute the googly amplitude we will need the off shell amplitudes with $n_+ = 1$. These amplitudes are obtained in [12]. There are two cases. The first case is when the first particle with moment p_1 is off shell and has positive helicity. The amplitude is

$$V_n(1+, 2-, \ldots, n-) = \frac{p_1^2}{\phi_2 \phi_n} \frac{1}{[2,3][3,4] \cdots [n-1,n]} \,. \tag{2.7}$$

The other case is when the off shell gluon has negative helicity. We relabel this gluon to be the first one and the amplitude is given as follows:

$$V_n(1-, 2-, \ldots, r+, \ldots, n-) = \frac{\phi_r^4 p_1^2}{\phi_2 \phi_n} \frac{1}{[2,3][3,4] \cdots [n-1,n]} . \tag{2.8}$$

We stress the fact that the above off shell amplitudes are proportional to p_1^2 and they vanish when p_1 is also on shell ($p_1^2 = 0$).

Now we compute the generic googly n-particle amplitude. The special case of which when the two positive helicity particles are adjacent was computed in [12]. The basic idea of the computation is the same. Now we label the two positive helicity particles to be 1 and r. As mentioned in the above, it was also proved in [12] that for any googly amplitude the contributing diagram will have only one 4 line MHV vertex. By using this result, the amplitude is computed by using the diagram decomposition as shown in figure 1.

For any given r, there are three kinds of diagrams as shown in figure 2, figure 3 and figure 4. They depend on whether the r-th gluon is in the second, third or the last blob.

All the 4 blob diagrams in these three kinds of diagrams can be computed by following the method used in [12]. By using the results in eqs. (2.7) and (2.8), the contribution

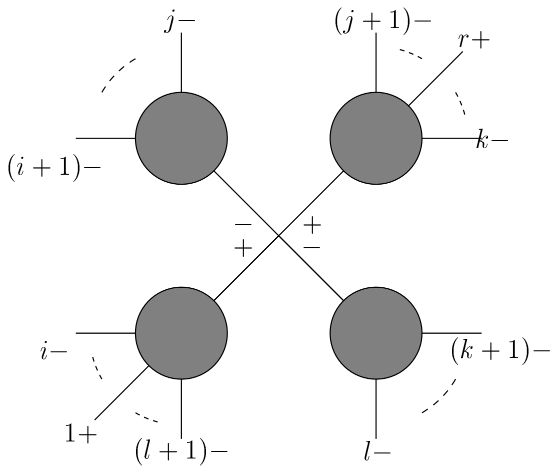

Figure 3: The diagram decomposition for the generic googly amplitude when $(j+1) \leq r \leq k$.

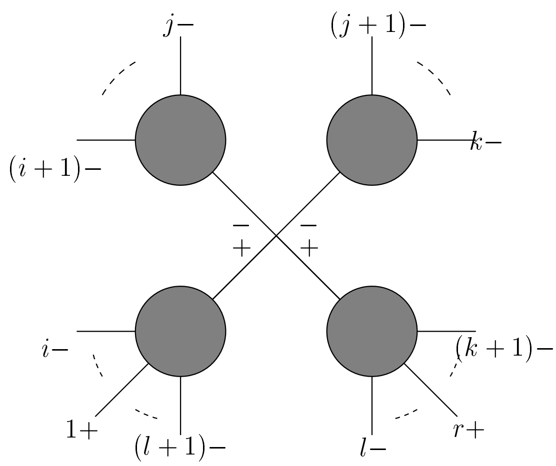

Figure 4: The diagram decomposition for the generic googly amplitude when $(k+1) \leq r \leq l$.

corresponding to figure 2 is

$$
\begin{aligned}
A_n^1 =& \sum_{i=1}^{r-1}\sum_{j=r}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n} \frac{\phi_1^4 p_{l+1,i}^2}{\phi_{l+1}\phi_i} \times \\
&\times \frac{1}{[l+1,l+2]\cdots[n-1,n][n,1][1,2]\cdots[i-1,i]} \times \\
&\times \frac{1}{p_{l+1,i}^2} \times \frac{\phi_r^4 p_{i+1,j}^2}{\phi_{i+1}\phi_j} \frac{1}{\prod_{t=i+1}^{j-1}[t,t+1]} \times \frac{1}{p_{i+1,j}^2} \times \frac{p_{j+1,k}^2}{\phi_{j+1}\phi_k} \frac{1}{\prod_{t=j+1}^{k-1}[t,t+1]} \times \\
&\times \frac{1}{p_{j+1,k}^2} \times \frac{p_{k+1,l}^2}{\phi_{k+1}\phi_l} \times \frac{1}{\prod_{t=k+1}^{l-1}[t,t+1]} \frac{1}{p_{k+1,l}^2} \times \frac{\langle V_3, V_4\rangle^3}{\langle V_1, V_2\rangle\langle V_2, V_3\rangle\langle V_4, V_1\rangle} \\
=& \frac{\phi_1^4\phi_r^4}{\prod_{t=1}^{n}[t,t+1]} \sum_{i=1}^{r-1}\sum_{j=r}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \times \\
&\times \frac{[k,k+1]}{\phi_k\phi_{k+1}} \frac{[l,l+1]}{\phi_l\phi_{l+1}} \frac{\langle V_3, V_4\rangle^3}{\langle V_1, V_2\rangle\langle V_2, V_3\rangle\langle V_4, V_1\rangle}
\end{aligned}
\tag{2.9}
$$

where[2]

$$
V_1 = \sum_{s=l+1}^{n+i} \lambda_s\phi_s\,, \qquad V_2 = \sum_{s=i+1}^{j} \lambda_s\phi_s\,, \tag{2.10}
$$

$$
V_3 = \sum_{s=j+1}^{k} \lambda_s\phi_s\,, \qquad V_4 = \sum_{s=k+1}^{l} \lambda_s\phi_s\,, \tag{2.11}
$$

and when $i \leq j$, we define $p_{i,j}$ as $p_{i,j} = \sum_{t=i}^{j} p_t$ as in [3], when $i > j$, we define $p_{i,j}$ as $p_{i,j} = \sum_{t=i}^{n} p_t + \sum_{t=1}^{j} p_t$.

[2]Here and below the summation like $\sum_{s=l+1}^{n+i}$ is understood as $\sum_{s=l+1}^{n} + \sum_{s=1}^{i}$.

The contribution corresponding to figure 3 and figure 4 is similar. The result are

$$A_n^2 = \frac{\phi_1^4\phi_r^4}{\prod_{t=1}^n [t,t+1]} \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} \sum_{k=r}^{n-1} \sum_{l=k+1}^{n} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \times$$
$$\times \frac{[k,k+1]}{\phi_k\phi_{k+1}} \frac{[l,l+1]}{\phi_l\phi_{l+1}} \frac{\langle V_2,V_4\rangle^4}{\langle V_1,V_2\rangle\langle V_2,V_3\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle} \tag{2.12}$$

and

$$A_n^3 = \frac{\phi_1^4\phi_r^4}{\prod_{t=1}^n [t,t+1]} \sum_{i=1}^{r-3} \sum_{j=i+1}^{r-2} \sum_{k=j+1}^{r-1} \sum_{l=r}^{n} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \times$$
$$\times \frac{[k,k+1]}{\phi_k\phi_{k+1}} \frac{[l,l+1]}{\phi_l\phi_{l+1}} \frac{\langle V_2,V_3\rangle^3}{\langle V_1,V_2\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle} \tag{2.13}$$

respectively.

The sum of A_n^i, $i=1,2,3$ can be written as

$$A_n = \frac{\phi_1^4\phi_r^4}{\prod_{t=1}^n [t,t+1]} \sum_{i=1}^{r-1} \sum_{l=\max\{i+3,r\}}^{n} \sum_{j=i+1}^{l-2} \sum_{k=j+1}^{l-1} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \times$$
$$\times \frac{[k,k+1]}{\phi_k\phi_{k+1}} \frac{[l,l+1]}{\phi_l\phi_{l+1}} \frac{\langle V_p,V_q\rangle^4}{\langle V_1,V_2\rangle\langle V_2,V_3\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle}, \tag{2.14}$$

where p and q in eq. (2.14) are the two indexes ($p=2,3, q=3,4, p\neq q$) which satisfy that neither V_p nor V_q includes $\lambda_r\phi_r$ as defined in eqs. (2.10) and (2.11).

As in [12], we can prove that the 4-fold summation in eq. (2.14) gives exactly the required result, i.e.

$$\sum_{i=1}^{r-1} \sum_{l=\max\{i+3,r\}}^{n} \sum_{j=i+1}^{l-2} \sum_{k=j+1}^{l-1} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \frac{[k,k+1]}{\phi_k\phi_{k+1}} \frac{[l,l+1]}{\phi_l\phi_{l+1}} \times$$
$$\times \frac{\langle V_p,V_q\rangle^4}{\langle V_1,V_2\rangle\langle V_2,V_3\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle} = \frac{[1,r]^4}{\phi_1^4\phi_r^4}. \tag{2.15}$$

The proof of this identity, eq. (2.15), will be given in appendix A. As in [12], it is proved by analyzing its pole terms and finding that all the pole terms are vanishing.

By using eq. (2.15) in eq. (2.14), we have

$$A_n(1+,2-,\ldots,r+,\ldots,n-) = \frac{[1,r]^4}{\prod_{i=1}^n [i,i+1]}. \tag{2.16}$$

This is the known result for the googly amplitude [4, 5]. It is the complex conjugate of the MHV amplitude, eq. (2.1), for Minkowski signature.

3. Some general discussions on the MHV diagrams in gauge theory

3.1 Parity transformation and the googly amplitude

As shown in the previous section, in our calculation of the googly amplitudes, we use the 3-line and 4-line MHV vertices only. The vertices with more than 4 lines don't appear in

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the contributing diagrams. A parity transformation will exchange the googly amplitudes with the MHV amplitudes. So in a parity invariant theory, once the off-shell continuation of 3 point and 4 point MHV amplitudes are given, the on-shell n point MHV amplitudes ($n > 4$) can be obtained from the googly amplitudes by parity transformation. The higher MHV amplitudes (and vertices) can't be chosen arbitrarily.

The parity invariance of the tree level amplitude is discussed in [14, 18] using string theory in twistor space [1]. Other related works are [16, 15, 17, 19].

We can make the parity transformation more precise. Denoting the parity transformation in Minkowski space by $\mathcal{P}$, we can choose its action on the momentum as follows:

$$\lambda_1(\mathcal{P}p) = -\tilde{\lambda}_2(p)\,, \qquad \lambda_2(\mathcal{P}p) = \tilde{\lambda}_1(p)\,. \tag{3.1}$$

This transformation satisfies our basic requirement of changing a quantity to its complex conjugate:

$$\langle \lambda(\mathcal{P}p_1), \lambda(\mathcal{P}p_2)\rangle = [\tilde{\lambda}(p_1), \tilde{\lambda}(p_2)]\,. \tag{3.2}$$

For the polarization vectors, we have

$$\epsilon^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda\rangle}, \qquad \epsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]}\,. \tag{3.3}$$

If we choose $\mu_1(\mathcal{P}p) = -\tilde{\mu}_2(p)$ and $\mu_2(\mathcal{P}p) = \tilde{\mu}_1(p)$, then we have $\epsilon^\mu(\mathcal{P}p, h) = P^\mu_\nu \epsilon^\nu(p, -h)$ (h is the helicity). From this we get

$$A(\mathcal{P}p_1, h_1; \ldots; \mathcal{P}p_n, h_n) = A(p_1, -h_1; \ldots;\, p_n,\, -h_n)\,. \tag{3.4}$$

So we have

$$\begin{aligned} A(p_1, +; \ldots, p_r, -; \ldots; p_s, -; \ldots; p_n, +) &= A(\mathcal{P}p_1, -; \ldots; \mathcal{P}p_r, +; \ldots; \mathcal{P}p_s, +, \ldots, \mathcal{P}p_n, -) \\ &= \frac{[\tilde{\lambda}(\mathcal{P}p_r), \tilde{\lambda}(\mathcal{P}p_s)]^4}{\prod_{i=1}^n [\tilde{\lambda}(\mathcal{P}p_i), \tilde{\lambda}(\mathcal{P}p_{i+1})]} \\ &= \frac{\langle \lambda(p_r), \lambda(p_s)\rangle^4}{\prod_{i=1}^n \langle \lambda(p_i), \lambda(p_{i+1})\rangle}\,. \end{aligned} \tag{3.5}$$

This shows that a parity transformation indeed exchange the googly amplitudes with the MHV amplitudes, as we expected.

3.2 The charge conjugation

Charge conjugation can be easily implemented in the CSW approach. In gauge theory, charge conjugate invariance gives the following identity for the tree-level partial amplitudes [6]:

$$A_n(p_1, h_1; p_2, h_2; \ldots; p_n, h_n) = (-1)^n A_n(p_n, h_n; \ldots; p_2, h_2; p_1, h_1)\,. \tag{3.6}$$

We note that the off shell MHV vertices also satisfies this identity. Because all the tree level amplitudes are computed from the MHV vertices in the framework of CSW, we could

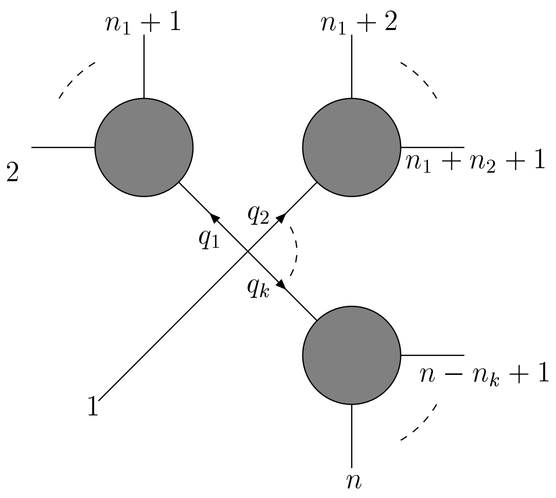

Figure 5: The diagram decomposition for the gauge amplitude $A(1,\ldots,n)$.

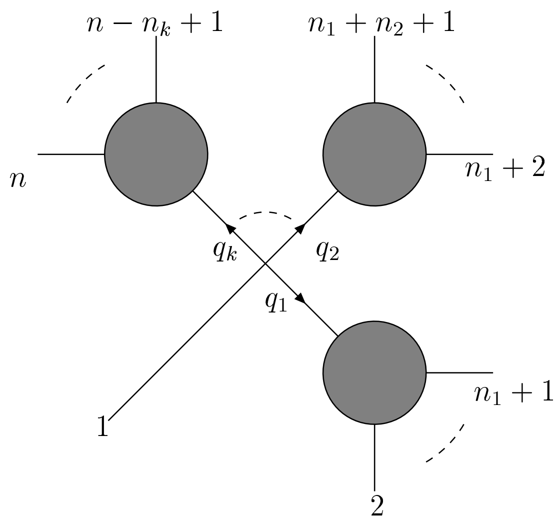

Figure 6: The diagram decomposition for the gauge amplitude $A(n,\ldots,1)$ which corresponds to the diagram decomposition in figure 5.

easily prove the above identity by doing charge conjugation operation on the "Feynman diagrams". We note that Eq. (3.6) has also been proved in [14] for the case when all particles are on-shell using the connected instanton in twistor string theory [1].

We will prove eq. (3.6) by mathematical induction. The case for $n=3$ can be easily proven case by case. Assuming that eq. (3.6) is true for all $k<n$ for both on-shell amplitudes and off-shell amplitudes, we will prove that it is also true for $k=n$.

The case when the amplitude $A_n(p_1,h_1;p_2,h_2;\ldots;p_n,h_n)$ has less than 2 gluons with negative helicity are trivial. Since

$$A_n(p_1,h_1;p_2,h_2;\ldots;p_n,h_n)=0=(-1)^n A_n(p_n,h_n;\ldots;p_2,h_2;p_1,h_1)\,. \tag{3.7}$$

When the amplitude $A_n(p_1,h_1;p_2,h_2;\ldots;p_n,h_n)$ is an MHV amplitude, the amplitude $A_n(p_n,h_n;\ldots;p_2,h_2;p_1,h_1)$ is an MHV amplitude, too. In this case we can easily prove eq. (3.6) by using eq. (2.1).

If the amplitude $A_n(p_1,h_1;p_2,h_2;\ldots;p_n,h_n)$ has more than 2 gluons with negative helicity, then the MHV diagrams has more than 1 MHV vertices [3]. We can use the diagram decomposition as in figure 5 to calculate this amplitude.[3] The helicities $h_{q_1},\ldots,h_{q_k}$ of the momenta $q_1,\ldots,q_k$ must satisfy the constraint that the vertex $V_{k+1}(p_1,h_1;q_1,h_{q_1};\ldots;q_k,h_{q_k})$ is an MHV vertex. For every diagram in this diagram decomposition, there is a unique corresponding diagram in the diagram decomposition used for calculating $A_n(p_n,h_n;\ldots;p_2,h_2;p_1,h_1)$ as in figure 6 with the same $h_{q_1},\ldots,h_{q_k}$. In fact this is a one-to-one correspondence between the diagrams in these two diagram decompositions.

[3]We note that when $n_+=0$, there are no contributing diagrams and the amplitude vanishes. Then it is trivial to find that eq. (3.6) is valid in this case.

By using the above diagram composition we have

$$
\begin{aligned}
A_n(p_1, h_1; p_2, h_2; \dots; p_n, h_n) = & \sum V_{k+1}(p_1, h_1; q_1, h_{q_1}; \dots; q_k, h_{q_k}) \times \frac{1}{q_1^2} \times \cdots \times \frac{1}{q_k^2} \times \\
& \times A_{n_1+1}(p_2, h_2; \dots; p_{n_1+1}, h_{n_1+1}; -q_1, -h_{q_1}) \times \\
& \times \cdots \times A_{n_k+1}(p_{n-n_k+1}, h_{n-n_k+1}; \dots; p_n, h_n; -q_k, -h_{q_k}) \,,
\end{aligned} \tag{3.8}
$$

where

$$
q_i = \sum_{j=n_1+n_2+\cdots+n_{i-1}+1}^{n_1+n_2+\cdots+n_i} p_j, \tag{3.9}
$$

for $1 \leq i \leq k$ and the summation in eq. (3.8) is over $n_1, \dots, n_k$ subject to the constrains $n_1 + \cdots + n_k = n - 1$, $n_i > 0$. There is also a summation over all possible helicity $h_{q_1}, \dots, h_{q_k}$ subject to the constraint that the vertex $V_{k+1}(p_1, h_1; q_1, h_{q_1}; \dots; q_k, h_{q_k})$ is an MHV vertex. In the above, the momentum p_1 can be off shell.

From the assumed result for all less multi-gluon amplitudes, we have

$$
\begin{aligned}
& A_n(p_1, h_1; p_2, h_2; \dots; p_n, h_n) = \\
& \quad = \sum (-1)^{k+1} V_{k+1}(p_1, h_1; q_k, h_{q_k}; \dots; q_1, h_{q_1}) \times \\
& \quad \times \frac{1}{q_1^2} \times \cdots \times \frac{1}{q_k^2} (-1)^{n_1+1} A_{n_1+1}(p_{n_1+1}, h_{n_1+1}; \dots; p_2, h_2; -q_1, -h_{q_1}) \times \\
& \quad \times \cdots \times (-1)^{n_k+1} A_{n_k+1}(p_n, h_n; \dots; p_{n-n_k+1}, h_{n-n_k+1}; -q_k, -h_{q_k}) \\
& \quad = \sum V_{k+1}(p_1, h_1; q_k, h_{q_k}; \dots, q_1, h_{q_1}) \frac{1}{q_1^2} \times \cdots \\
& \quad \times \frac{1}{q_k^2} A_{n_1+1}(p_{n_1+1}, h_{n_1+1}; \dots; p_2, h_2; -q_1, -h_{q_1}) \times \cdots \\
& \quad \times A_{n_k+1}(p_n, h_n; \dots; p_{n-n_k+1}, h_{n-n_k+1}; -q_k, -h_{q_k}) \times \\
& \quad \times (-1)^{k+1+n_1+n_2+\cdots+n_k+k} \\
& \quad = (-1)^n A_n(p_n, h_n; \dots; p_2, h_2; p_1, h_1) \,.
\end{aligned} \tag{3.10}
$$

The degenerate case when some n_i's equal to 1 is also correctly included in the previous equation. This completes our proof of eq. (3.6).

3.3 The dual Ward identity

The dual Ward identity is [5]

$$
\sum_{\sigma \in Z_{n-1}} A(\sigma(1), \dots, \sigma(n-1), n) = 0 \,. \tag{3.11}
$$

where the summation σ is over all of the cyclic permutation of $1, \dots, n-1$ and the position of n is held fixed to be the last one by using the cyclic symmetry of the amplitude. This identity reflect the decoupling of the U(1) degree of freedom and links the factorization of the partial amplitudes to the factorization of the full amplitudes [6]. This has been discussed in [14]. It is also proved in [36] that the dual Ward identity is valid for (on-shell) MHV amplitudes. We will show that the above dual Ward identity is valid for all tree-level amplitudes in the framework of CSW.

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Firstly, we prove that the dual Ward identity is true for the off shell MHV vertices. By assuming that the two gluons with negative helicity are g_p and g_q, we must prove the following:

$$\sum_{\sigma\in Z_{n-1}} V(\sigma(1),\ldots,\sigma(n-1),n) = \langle p,q\rangle^4 \sum_{\sigma\in Z_{n-1}} \frac{1}{\langle\sigma(1),\sigma(2)\rangle\cdots\langle\sigma(n-1),n\rangle\langle n,\sigma(1)\rangle} = 0\,. \tag{3.12}$$

In order to prove this last equality, we note first the following relation:

$$\langle i,j\rangle = \lambda_{i1}\lambda_{j2} - \lambda_{i2}\lambda_{j1} = \left(\frac{\lambda_{i1}}{\lambda_{i2}} - \frac{\lambda_{j1}}{\lambda_{j2}}\right)\lambda_{i2}\lambda_{j2}\,. \tag{3.13}$$

Setting $\psi_i = \lambda_{i1}/\lambda_{i2}$, we have

$$\begin{aligned}\sum_{\sigma\in Z_{n-1}} &\frac{1}{\langle\sigma(1),\sigma(2)\rangle\ \cdots\langle\sigma(n-1),n\rangle\ \langle n,\sigma(1)\rangle} = \\ &= \frac{1}{(\prod_{i=1}^n \lambda_{i2})^2} \times \sum_{\sigma\in Z_{n-1}} \frac{1}{(\psi_{\sigma(1)}-\psi_{\sigma(2)})\cdots(\psi_{\sigma(n-1)}-\psi_n)(\psi_n-\psi_{\sigma(1)})}\,.\end{aligned} \tag{3.14}$$

So what we need to prove is the following identity:

$$\sum_{\sigma\in Z_{n-1}} \frac{1}{(\psi_{\sigma(1)}-\psi_{\sigma(2)})\cdots(\psi_{\sigma(n-1)}-\psi_n)(\psi_n-\psi_{\sigma(1)})} = 0\,, \tag{3.15}$$

for arbitrary ψ_i's.

We will prove eq. (3.15) by mathematical induction. The case for $n=3$ can be easily checked to be valid. Assuming that it is valid for $k=n-1$, we will show that it is valid for $k=n$.

We will first prove that the l.h.s. is a constant by showing that all the pole terms are vanishing. When $\psi_i\to\infty$ $(1\le i\le n+1)$, it is evident that the pole terms are vanishing. So we only need to consider the finite pole terms. The possible finite pole terms appear when $\psi_j=\psi_n$, $1\le j\le n-1$, or $\psi_i=\psi_{i+1}$, $1\le i\le n-2$, or $\psi_{n-1}=\psi_1$.

When $\psi_j=\psi_n$, $1\le j\le n-1$, the pole terms in eq. (3.15) are from the following two cyclic permutations:

$$\sigma_{j-1}(1,\ldots,n-1) = (j,j+1,\ldots,n-1,1,2,\ldots,j-2,j-1)\,, \tag{3.16}$$

and

$$\sigma_j(1,\ldots,n-1) = (j+1,j+2,\ldots,n-1,1,2,\ldots,j-1,j)\,. \tag{3.17}$$

These give 2 pole terms:

$$\begin{aligned}&\frac{1}{\psi_j-\psi_{j+1}}\frac{1}{\psi_{j+1}-\psi_{j+2}}\cdots\frac{1}{\psi_{j-1}-\psi_n}\frac{1}{\psi_n-\psi_j}+ \\ &+\frac{1}{\psi_n-\psi_{j+1}}\frac{1}{\psi_{j+1}-\psi_{j+2}}\cdots\frac{1}{\psi_{j-1}-\psi_j}\frac{1}{\psi_j-\psi_n} = \\ &\quad= \frac{1}{\psi_n-\psi_j}\left[\frac{1}{\psi_j-\psi_{j+1}}\frac{1}{\psi_{j+1}-\psi_{j+2}}\cdots\frac{1}{\psi_{j-1}-\psi_n} - (\psi_n\leftrightarrow\psi_j)\right].\end{aligned} \tag{3.18}$$

The residues in the bracket vanish by taking $\psi_j=\psi_n$.

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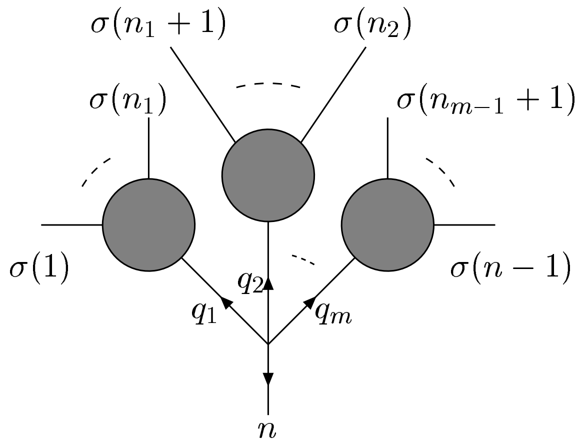

Figure 7: The decomposition of the amplitude $A(\sigma(1),\ldots,\sigma(n-1),n)$ in the proof of the dual Ward identity.

For $\psi_i = \psi_{i+1}$, $1 \le i \le n-2$, the only cyclic permutation which doesn't give a pole term is:

$$\sigma_i(1,2,\ldots,n-1) = (i+1, i+2, \ldots, n-1, 1, 2, \ldots, i-1, i). \tag{3.19}$$

The residues are

$$\sum_{\sigma\neq\sigma_i} \frac{1}{\psi_{\sigma(1)}-\psi_{\sigma(2)}}\frac{1}{\psi_{\sigma(2)}-\psi_{\sigma(3)}}\cdots\frac{1}{\psi_{i-1}-\psi_{i+1}}\frac{1}{\psi_{i+1}-\psi_{i+2}}\cdots\frac{1}{\psi_{\sigma(n-1)}-\psi_n}\frac{1}{\psi_n-\psi_{\sigma(1)}}. \tag{3.20}$$

One easily convinces oneself that the above summation is actually the l.h.s. of eq. (3.15) with the deletion of ψ_i and so it sums to zero by the assumption of our mathematical induction. This proves that the residues for $\psi_i = \psi_{i+1}$ vanish. The case for $\psi_{n-1} = \psi_1$ can be proved by the same method.

So we proved that all the finite pole terms are vanishing and we conclude that the l.h.s. must be a constant. This constant can only be zero because it is homogeneous (with negative degree) in ψ_i's. This completes our proof of eq. (3.15).

By using the above result for dual Ward identity for the off-shell MHV vertices, one can easily prove the general dual Ward identity, eq. (3.11). The strategy is to write each amplitude $A(\sigma(1),\ldots,\sigma(n-1),n)$ as a sum over all possible MHV vertices connected with the n-th gluon as shown in figure 7. For a fixed MHV vertex, the summation over all possible cyclic permutations are decomposed as a sum over cyclic permutations of q_1, q_2, $\ldots$, q_m.[4] Each sum over cyclic permutations of q_1, q_2, $\ldots$, q_m is zero by using the dual Ward identity for MHV vertex and the general dual Ward identity then follows.

4. The googly amplitudes with fermions

The proposal of [3] can also be extended to gauge theory with fermions (see also [21]). Although some of the MHV (and googly) fermionic amplitudes can be easily obtained by

[4]This statement should be proved rigorously. We will not try to spell out the full details of the proof here.

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supersymmetric Ward identities [30, 31, 6] we think it is still worthy to compute these amplitudes directly because the general non-MHV (googly) ammplitudes cannot be determined in terms of amplitudes with gluons only and should be computed separately. In this section we will compute the googly amplitudes with fermions by extending the CSW rules with MHV vertices. We will consider only the simplest case of a single quark-anti-quark pair. The general cases, including the supersymmetric case with gluinos, will be discussed in a separate publication [33]. We also note that some generic non-MHV fermionic amplitudes were also computed in [21, 32].

For the case of a single quark-anti-quark pair, the MHV vertices are as follows (only the I-th gluon has negative helicity):

$$A(\Lambda_q^+, g_1^+, \ldots, g_I^-, \ldots, g_n^+, \Lambda_{\bar{q}}^-) = -\frac{\langle q, I\rangle\langle \bar{q}, I\rangle^3}{\langle q, 1\rangle\langle 1, 2\rangle \cdots \langle n, \bar{q}\rangle\langle \bar{q}, q\rangle}\,, \tag{4.1}$$

$$A(\Lambda_q^-, g_1^+, \ldots, g_I^-, \ldots, g_n^+, \Lambda_{\bar{q}}^+) = \frac{\langle q, I\rangle^3\langle \bar{q}, I\rangle}{\langle q, 1\rangle\langle 1, 2\rangle \cdots \langle n, \bar{q}\rangle\langle \bar{q}, q\rangle}\,, \tag{4.2}$$

by denoting the quark with helicity $\pm$ as $\Lambda_q^\pm$ and the anti-quark as $\Lambda_{\bar{q}}^\pm$. Gluons are denoted as $g_i^\pm$ in an obvious notation. What we want to do is to reproduce the following googly amplitudes (only the I-th gluon has positive helicity):

$$A(\Lambda_q^+, g_1^-, \ldots, g_I^+, \ldots, g_n^-, \Lambda_{\bar{q}}^-) = \frac{[q, I]^3[\bar{q}, I]}{[q, 1][1, 2] \cdots [n, \bar{q}][q, \bar{q}]}\,, \tag{4.3}$$

$$A(\Lambda_q^-, g_1^-, \ldots, g_I^+, \ldots, g_n^-, \Lambda_{\bar{q}}^+) = -\frac{[q, I][\bar{q}, I]^3}{[q, 1][1, 2] \cdots [n, \bar{q}][q, \bar{q}]}\,. \tag{4.4}$$

We use the same off shell continuation as given in [3] (see also section 2) for off-shell momenta lines. The propagator for both gluon and gluino internal lines is just $1/p^2$, as explained in [21].

We first calculate the amplitudes with a quark-anti-quark pair when all gluons have negative helicity and only one particle is off-shell. By using the same method as in [12], we find that all vertices in the contribution diagrams are 3-line MHV vertices. When all the 3 particles are off shell, the MHV amplitudes with a quark-anti-quark pair can be written as:

$$A(\Lambda_q^+, g_1^-, \Lambda_{\bar{q}}^-) = \frac{\langle 1, \bar{q}\rangle^2}{\langle \bar{q}, q\rangle} = \langle q, 1\rangle = \langle 1, \bar{q}\rangle = \langle \bar{q}, q\rangle\,, \tag{4.5}$$

$$A(\Lambda_q^-, g_1^-, \Lambda_{\bar{q}}^+) = -\frac{\langle q, \bar{1}\rangle^2}{\langle \bar{q}, q\rangle} = -\langle q, 1\rangle = -\langle 1, \bar{q}\rangle = -\langle \bar{q}, q\rangle\,. \tag{4.6}$$

The amplitudes $A(\Lambda_q^\pm, g_1^-, \ldots, g_n^-, \Lambda_{\bar{q}}^\mp)$ when $\Lambda_q^\pm$ is off shell are given as follows:

$$A(\Lambda_q^+, g_1^-, \ldots, g_n^-, \Lambda_{\bar{q}}^-) = \frac{p_q^2}{\phi_1}\frac{1}{[1, 2][2, 3] \cdots [n-1, n][n, \bar{q}]}, \tag{4.7}$$

$$A(\Lambda_q^-, g_1^-, \ldots, g_n^-, \Lambda_{\bar{q}}^+) = -\frac{p_q^2}{\phi_1}\frac{1}{[1, 2][2, 3] \cdots [n-1, n][n, \bar{q}]}\,, \tag{4.8}$$

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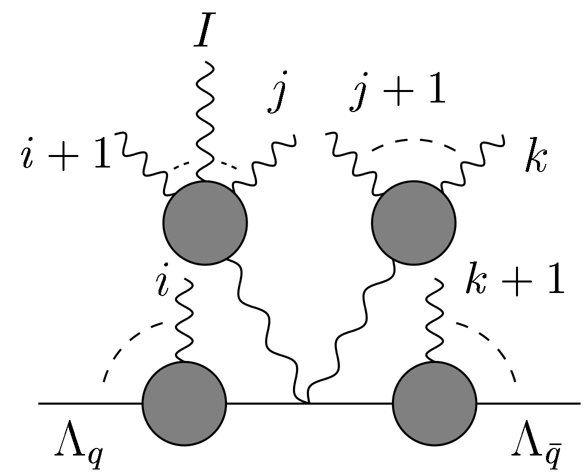

Figure 8: A sample diagram of 4-line MHV vertex with the quark-anti-quark pair.

and

$$A(\Lambda_q^+, g_1^-, \ldots, g_n^-, \Lambda_{\bar q}^-) = \frac{p_{\bar q}^2 \phi_q^2}{\phi_n} \frac{1}{[q,1][1,2][2,3]\cdots[n-1,n]}, \tag{4.9}$$

$$A(\Lambda_q^-, g_1^-, \ldots, g_n^-, \Lambda_{\bar q}^+) = -\frac{p_{\bar q}^2 \phi_q^2}{\phi_n} \frac{1}{[q,1][1,2][2,3]\cdots[n-1,n]}, \tag{4.10}$$

when the anti-quark $\Lambda_{\bar q}^{\pm}$ is off-shell. We will not give the proof of the above formulas here.

When the off shell particle is g_i, the result is

$$\begin{aligned} A(\Lambda_q^+, g_1^-, \ldots, g_n^-, \Lambda_{\bar q}^-) &= \frac{p_i^2 \phi_q^3 \phi_{\bar q}}{\phi_{i-1}\phi_{i+1}} \times \\ &\times \frac{1}{[q,1][1,2]\cdots[i-2,i-1][i+1,i+2]\cdots[n,\bar q][\bar q,q]}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} A(\Lambda_q^-, g_1^-, \ldots, g_n^-, \Lambda_{\bar q}^+) &= -\frac{p_i^2 \phi_q^3 \phi_{\bar q}}{\phi_{i-1}\phi_{i+1}} \times \\ &\times \frac{1}{[q,1][1,2]\cdots[i-2,i-1][i+1,i+2]\cdots[n,\bar q][\bar q,q]}. \end{aligned} \tag{4.12}$$

Here and in the remaining part of this section the index 0 refers to Λ_q and the index $n+1$ refers to $\Lambda_{\bar q}$.

Now we begin to compute the googly amplitudes with one quark-anti-quark pair. Using the same method used in [12], one can find that in every contributing diagram there is only one vertex with 4 lines, all other vertices are with 3 lines. For the amplitude $A(\Lambda_q^+, g_1^-, \ldots, g_I^+, \ldots, g_n^-, \Lambda_{\bar q}^-)$, one sample diagram was shown in figure 8. By using the known results for the amplitudes with exactly one off shell particle given previously in eqs. (2.7)–(2.8) and eqs. (4.7)–(4.12), this gives:

$$\begin{aligned} &\frac{\phi_q^2 \phi_I^4}{[q,1][1,2]\cdots[n-1,n][n,\bar q]} \sum_{i=0}^{I-1} \sum_{j=I}^{n-1} \sum_{k=j+1}^{n} \frac{[i,i+1]}{\phi_i \phi_{i+1}} \frac{[j,j+1]}{\phi_j \phi_{j+1}} \times \\ &\qquad \times \frac{[k,k+1]}{\phi_k \phi_{k+1}} \frac{-\langle V_1,V_3\rangle \langle V_4,V_3\rangle^3}{\langle V_1,V_2\rangle\langle V_2,V_3\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle}, \end{aligned} \tag{4.13}$$

where

$$V_1 = \sum_{s=0}^{i} \lambda_s \phi_s \,, \qquad V_2 = \sum_{s=i+1}^{j} \lambda_s \phi_s \,, \tag{4.14}$$

$$V_3 = \sum_{s=j+1}^{k} \lambda_s \phi_s \,, \qquad V_4 = \sum_{s=k+1}^{n+1} \lambda_s \phi_s \,. \tag{4.15}$$

The complete result is

$$\begin{aligned} A = & \frac{\phi_q^2 \phi_I^4}{[q,1][1,2]\cdots[n-1,n][n,\bar{q}]} \sum_{i=0}^{I-1} \sum_{k=\max\{I,i+2\}}^{n} \sum_{j=i+1}^{k-1} \frac{[i,i+1]}{\phi_i \phi_{i+1}} \times \\ & \times \frac{[j,j+1]}{\phi_j \phi_{j+1}} \frac{[k,k+1]}{\phi_k \phi_{k+1}} \frac{-\langle V_1, V_p\rangle \langle V_4, V_p\rangle^3}{\langle V_1, V_2\rangle \langle V_2, V_3\rangle \langle V_3, V_4\rangle \langle V_4, V_1\rangle} + \\ & + \frac{\phi_q^3 \phi_{\bar{q}} \phi_I^4}{[q,1][1,2]\cdots[n-1,n][n,\bar{q}][\bar{q},q]} \sum_{i=0}^{I-1} \sum_{l=max\{I,i+3\}}^{n} \sum_{j=i+1}^{l-2} \sum_{k=j+1}^{l-1} \frac{[i,i+1]}{\phi_i \phi_{i+1}} \times \\ & \times \frac{[j,j+1]}{\phi_j \phi_{j+1}} \frac{[k,k+1]}{\phi_k \phi_{k+1}} \frac{[l,l+1]}{\phi_l \phi_{l+1}} \frac{\langle \tilde{V}_r, \tilde{V}_s\rangle^4}{\langle \tilde{V}_1, \tilde{V}_2\rangle \langle \tilde{V}_2, \tilde{V}_3\rangle \langle \tilde{V}_3, \tilde{V}_4\rangle \langle \tilde{V}_4, \tilde{V}_1\rangle} \,, \end{aligned} \tag{4.16}$$

where

$$\tilde{V}_1 = \sum_{s=0}^{i} \lambda_s \phi_s + \sum_{s=l+1}^{n+1} \lambda_s \phi_s \,, \qquad \tilde{V}_2 = \sum_{s=i+1}^{j} \lambda_s \phi_s \,, \tag{4.17}$$

$$\tilde{V}_3 = \sum_{s=j+1}^{k} \lambda_s \phi_s \,, \qquad \tilde{V}_4 = \sum_{s=k+1}^{l} \lambda_s \phi_s \,. \tag{4.18}$$

The p in eq. (4.16) is the index ($p = 2, 3$) which satisfies that V_p doesn't include $\lambda_I \phi_I$ as defined in eqs. (4.14) and (4.15) and the r and s in eq. (4.16) are the two indices ($r = 2, 3$, $s = 3, 4$, $r \neq s$) which satisfy that neither $\tilde{V}_r$ nor $\tilde{V}_s$ includes $\lambda_I \phi_I$ as defined in eqs. (4.17) and (4.18). The summations in eq. (4.16) can be done by using identities involving λ_i as in the gluon googly amplitude. After doing this summation we obtained the correct googly amplitude with a single quark-anti-quark pair as given in eqs. (4.3) and (4.4). The details for the relevant identities and also the extension to multi-pair of quark-anti-quark pairs and gluinos will be given in a separate publication [33].

5. The MHV diagrams for gravity

Encouraged by the success of the CSW approach to gauge theory, it is natural to ask if a similar approach to gravity exists. We expected so also because there exist some close relations between the gravity amplitudes and the gauge amplitudes, the famous KLT relations [35].

We use the same off-shell continuation $\lambda_a = p_{a\dot{a}} \tilde{\eta}^{\dot{a}}$ as in [3]. The off shell MHV vertex with 3 gravitons is [34]

$$V_3^{\text{graviton}}(1+, 2-, 3-) = \left(\frac{\langle 2,3\rangle^3}{\langle 1,2\rangle \langle 3,1\rangle} \right)^2 , \tag{5.1}$$

and the propagator for the graviton with momentum p is still $1/p^2$.

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As in gauge theory, we first compute $A(1+, 2-, \ldots, n-)$ when only one particle is off-shell. We will give a formula for $A(1+, 2-, \ldots, n-)$ when only the momentum p_i is off-shell. This amplitude vanishes when all momenta are on shell. So we expect that it is proportional to p_i^2. This turns out to be true for gravity. We note that the vanishing for the 4-particle and 5-particle on-shell amplitudes were also obtained in [24].

The diagrams which contribute to $A(1+, 2-, \ldots, n-)$ include the MHV vertices with 3 lines only. By using momentum conservation, one can show that the MHV vertex $V_3(1+, 2-, 3-)$ is actually polynomial in λ:

$$V_3(1+, 2-, 3-) = \langle \lambda_1, \lambda_2 \rangle^2 = \langle \lambda_2, \lambda_3 \rangle^2 = \langle \lambda_3, \lambda_1 \rangle^2 . \tag{5.2}$$

In order to present the rules for the computation of $A(1+, 2-, \ldots, n-)$, we will need some basic concepts and notations from graph theory. By a graph Γ, we mean two set: the vertex set V and the edge set $E \subseteq \{e_{ij} | i, j \in V, i \neq j\}$. We will use the undirected graph only. This means that $e_{ij} = e_{ji}$ for all $i, j \in V$. We will denote the vertex set of graph Γ as $V(\Gamma)$ and denote the edge set of graph Γ as $E(\Gamma)$. A connected graph not containing any cycles is called a tree. Two vertex i, j are adjacent if there is a edge e_{ij} in $E(\Gamma)$.

We relabel the off-shell graviton to be the first one. The formula for the amplitudes is:

$$A(p_1, h_1; p_2, h_2; \ldots; p_n, h_n) = p_1^2 \sum_{\Gamma, V(\Gamma) = S(n)} P(\Gamma) , \tag{5.3}$$

where $h_i = \pm 2$ is the helicity of the graviton whose momentum is p_i and Γ is undirected tree whose vertex set is $V(\Gamma) = S(n) \equiv \{2, 3, \ldots, n\}$ and $P(\Gamma)$ can be obtained from the following rules:

- For every vertex i in $V(\Gamma)$, there is a factor $p(i) = \phi_i^{2h_i}$. It means that if the graviton with momentum p_i has positive helicity, the factor is ϕ_i^4; otherwise the factor is ϕ_i^{-4}. We stress the fact that there is only one positive helicity graviton.

- For every edge e_{ij} in $E(\Gamma)$ there is a factor $p(e_{ij})$:

$$p(e_{ij}) = \phi_i^2 \, \phi_j^2 \, \frac{\langle i, j \rangle}{[i, j]} . \tag{5.4}$$

$P(\Gamma)$ is the product of above factors:

$$P(\Gamma) = \prod_{i \in V(\Gamma)} p(i) \prod_{e_{ij} \in E(\Gamma)} p(e_{ij}) . \tag{5.5}$$

By using the above rules we have

$$A(p_1, h_1, \ldots, p_n, h_n) = p_1^2 \prod_{i=2}^{n} \phi_i^{2h_i} \sum_{\Gamma, V(\Gamma) = S(n)} \prod_{e_{ij} \in E(\Gamma)} p(e_{ij}) . \tag{5.6}$$

The summation in eq. (5.6) is over all of the inequivalent undirected trees with vertex set $S(n)$. The result can be proved by mathematical induction. The detail is relegated to appendix B.

The on-shell 4-graviton MHV amplitude is [34]:

$$A_4(1+,2+,3-,4-) = \frac{\langle 3,4\rangle^8}{\langle 1,2\rangle\langle 1,3\rangle\langle 1,4\rangle\langle 2,3\rangle\langle 2,4\rangle\langle 3,4\rangle}\frac{[3,4]}{\langle 1,2\rangle}\,. \tag{5.7}$$

We note that a straightforward off-shell continuation $\lambda_a = p_{a\dot{a}}\tilde{\eta}^{\dot{a}}$ is not possible for A_4 because of the existence of the "anti-holomorphic term" $[3,4]$. By using the on-shell condition, like $p_3^2 = p_4^2 = 0$, and momentum conservation, one can write this term in various forms which are equivalent on shell and include the "holomorphic terms" and the momenta product only.

For example, we can use $\langle 3,4\rangle[3,4] = 2p_3{\cdot}p_4$ to write $[3,4]$ as $2p_3{\cdot}p_4/\langle 3,4\rangle$. On the other hand, using the on-shell condition $p_i^2 = 0, i = 1,\ldots,4$ and the momentum conservation, we have $2p_3 \cdot p_4 = p_{34}^2 = 2p_1 \cdot p_2$. We can use these relations to give two more way of writing the "non-holomorphic" expression $[3,4]$.

We can also use $\langle 1,2\rangle[2,3] + \langle 1,4\rangle[4,3] = 0$ to write

$$\frac{[3,4]}{\langle 1,2\rangle} = \frac{[2,3]}{\langle 1,4\rangle} = \frac{p_{23}^2}{\langle 2,3\rangle\langle 1,4\rangle}\,, \tag{5.8}$$

or use $\langle 1,3\rangle[3,4] + \langle 1,2\rangle[2,4] = 0$ to write

$$\frac{[3,4]}{\langle 1,2\rangle} = \frac{[4,2]}{\langle 1,3\rangle} = \frac{p_{13}^2}{\langle 1,3\rangle\langle 4,2\rangle}\,. \tag{5.9}$$

One can try to write all the possible (on shell) equivalent forms for the 4-graviton MHV amplitude. There are about 9 different forms which satisfy the obvious symmetry. Nevertheless we have not been able to obtain the correct 5-graviton amplitude by using any one of these forms or one of their linear combination. It is safe to conclude that a naive extension of the CSW approach to gravity failed (see also [24]). New ingredient must be introduced in order to develop a CSW like rules for gravity. The similar connection between conformal supergravity and twistor string theory discussed in the recent paper [26] may offer some clues.

To extend the CSW approach further, we may change the 3-particle vertex. A possible "MHV vertex" for charged particle is

$$\begin{aligned} A(1+,2-,3-) &= \left(\frac{\langle 2,3\rangle^3}{\langle 1,2\rangle\langle 3,1\rangle}\right)^{2s-1} \\ &= \langle 1,2\rangle^{2s-1} = \langle 2,3\rangle^{2s-1} = \langle 3,1\rangle^{2s-1}\,, \end{aligned} \tag{5.10}$$

where s is a positive integer. By using this vertex and the CSW rules, one can also compute $A(1+,2-,3-,4-)$ and the result is

$$\begin{aligned} A(1+,2-,3-,4-) &= (\phi_2\phi_3\phi_4)^{-(2s-1)} \times \left(\frac{\langle 2,3\rangle^{2s-2}}{[2,3]}\langle\lambda_{23},\lambda_{p4}\rangle^{2s-1}\right. \\ &\qquad\left. + \frac{\langle 3,4\rangle^{2s-2}}{[3,4]}\langle\lambda_{p2},\lambda_{34}\rangle^{2s-1}\right). \end{aligned} \tag{5.11}$$

By using the following equation:

$$\frac{\langle \lambda_{23}, \lambda_{p4} \rangle}{[2,3]} + \frac{\langle \lambda_{p2}, \lambda_{34} \rangle}{[3,4]} = 0\,, \tag{5.12}$$

we have

$$\begin{aligned} A(1+,2-,3-,4-) &= \frac{1}{[2,3]^{2s-1}} \left[(p_2 \cdot p_3)^{2s-2} - (p_3 \cdot p_4)^{2s-2} \right] \times \\ &\quad \times \langle \lambda_{23}, \lambda_{p4} \rangle^{2s-1} (\phi_2 \phi_3 \phi_4)^{-(2s-1)}\,. \end{aligned} \tag{5.13}$$

So when $s > 1$ this amplitudes doesn't vanish for generic $\lambda_i, \tilde{\lambda}_i (1 \le i \le 4)$. This indicates that the CSW approach can't be arbitrarily extended to include higher derivative theories. It remains to discover the hidden principle behind the success of the CSW approach to gauge theory and its coupling to fermions, at least for tree level calculations.

Acknowledgments

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A. The proof of eq. (2.15)

In this appendix we will prove eq. (2.15). As in [12], we can use a $SL(2, \mathbf{C})$ transformation and a rescaling of $\tilde{\eta}$ to choose $\tilde{\eta}^1 = 0$ and $\tilde{\eta}^2 = 1$. Then we have

$$\phi_i = \tilde{\lambda}_{i2} \tag{A.1}$$

and

$$\frac{[i,j]}{\phi_i \phi_j} = \frac{\tilde{\lambda}_{i1}\tilde{\lambda}_{j2} - \tilde{\lambda}_{i2}\tilde{\lambda}_{j1}}{\tilde{\lambda}_{i2}\tilde{\lambda}_{j2}} = \frac{\tilde{\lambda}_{i1}}{\tilde{\lambda}_{i2}} - \frac{\tilde{\lambda}_{j1}}{\tilde{\lambda}_{j2}}\,. \tag{A.2}$$

If we do a rescaling of $\tilde{\lambda}_{i1}$ by $\tilde{\lambda}_{i2}$, i.e. by defining $\varphi_i = \frac{\tilde{\lambda}_{i1}}{\tilde{\lambda}_{i2}}$, and also do a rescaling of λ_{ia} by $1/\tilde{\lambda}_{i2}$, then eq. (2.15) becomes:

$$\begin{aligned} F(\varphi) &= \sum_{i=1}^{r-1} \sum_{l=\max\{i+3,r\}}^{n} \sum_{j=i+1}^{l-2} \sum_{k=j+1}^{l-1} (\varphi_i - \varphi_{i+1})(\varphi_j - \varphi_{j+1}) \times \\ &\quad \times (\varphi_k - \varphi_{k+1})(\varphi_l - \varphi_{l+1}) \frac{\langle V_p, V_q \rangle^4}{\langle V_1, V_2 \rangle \langle V_2, V_3 \rangle \langle V_3, V_4 \rangle \langle V_4, V_1 \rangle} \\ &= (\varphi_1 - \varphi_r)^4\,. \end{aligned} \tag{A.3}$$

where

$$V_1 = \sum_{s=l+1}^{n+i} \lambda_s \,, \qquad V_2 = \sum_{s=i+1}^{j} \lambda_s \,, \tag{A.4}$$

$$V_3 = \sum_{s=j+1}^{k} \lambda_s \,, \qquad V_4 = \sum_{s=k+1}^{l} \lambda_s \,. \tag{A.5}$$

There are also two constraints:

$$V_1 + V_2 + V_3 + V_4 = \sum_{l=1}^{n} \lambda_l = 0 \,, \tag{A.6}$$

$$\sum_{l=1}^{n} \lambda_i \, \varphi_i = 0 \,, \tag{A.7}$$

from momentum conservation.

From eq. (A.6) and eq. (A.7) we can solve λ_1 and λ_r in terms of the rest λ_i's and all φ_j's as:

$$\lambda_1 = - \sum_{2 \le j \le n, j \ne r} \frac{\varphi_j - \varphi_r}{\varphi_1 - \varphi_r} \lambda_j \,, \tag{A.8}$$

$$\lambda_r = \sum_{2 \le j \le n, j \ne r} \frac{\varphi_j - \varphi_1}{\varphi_1 - \varphi_r} \lambda_j \,. \tag{A.9}$$

By using the above result, the left hand side of eq. (A.3) can be considered as a function of λ_j $(2 \le j \le n,\, j \ne r)$ and all φ_j. As a function of φ we will show that it is independent of φ_j for $2 \le j \le n$ and $j \ne r$.

First we show that there is no pole terms for $\varphi_s \to \infty, 2 \le s \le n-1, s \ne r$. We will show that every term in the l.h.s. of eq. (A.3) will not tend to infinity when $\varphi_s \to \infty$. We denote $(\varphi_i - \varphi_{i+1})(\varphi_j - \varphi_{j+1})(\varphi_k - \varphi_{k+1})(\varphi_l - \varphi_{l+1})$ in eq. (A.3) as $F(i,j,k,l)$ and $\langle V_p, V_q \rangle^4 / (\langle V_1, V_2 \rangle \langle V_2, V_3 \rangle \langle V_3, V_4 \rangle \langle V_4, V_1 \rangle)$ as $A_4(V_1, V_2, V_3, V_4)$.

When $\varphi_s \to \infty$, λ_1 and λ_r will grow as φ_s and other λ_i's does not depend on φ_s. V_1 and the V_t which includes λ_r as defined in eqs. (A.4) and (A.5) will tend to infinity as fast as φ_s. V_p and V_q are independent of φ_s. So the numerator of $A_4(V_1, V_2, V_3, V_4)$ is independent of φ_s.

Now we prove that $\langle V_1, V_t \rangle$ grows as φ_s. From eq. (A.8) and eq. (A.9), we can write λ_1 and λ_r as $\lambda_1 = a_{1s}\varphi_s\lambda_s + \mu_{1s}$ and $\lambda_r = a_{rs}\varphi_s\lambda_s + \mu_{rs}$ respectively, where μ_{1s} and μ_{rs} is independent on φ_s. Then $\langle V_1, V_t \rangle = \varphi_s \langle \lambda_s, a_{1s}\mu_{rs} - a_{rs}\mu_{1s} \rangle + O(\varphi_s^0)$ will grow as φ_s for generic $\varphi_i, \lambda_j, 1 \le j \le n, j \ne r$.

The factor $F(i,j,k,l)$ grows as φ_s^2 at most. Also one can check case by case and find the denominator of $A_4(V_1, V_2, V_3, V_4)$ grows as φ_s^2 at least.[5] By combing all these results one sees that there is no pole terms for $F(\varphi)$ as $\varphi_s \to \infty$.

[5]We note that when one of V_p and V_q includes only the term λ_s, the denominator will grows as φ_s^2 because $\langle V_1, \lambda_s \rangle$ and $\langle V_t, \lambda_s \rangle$ are independent of φ_s.

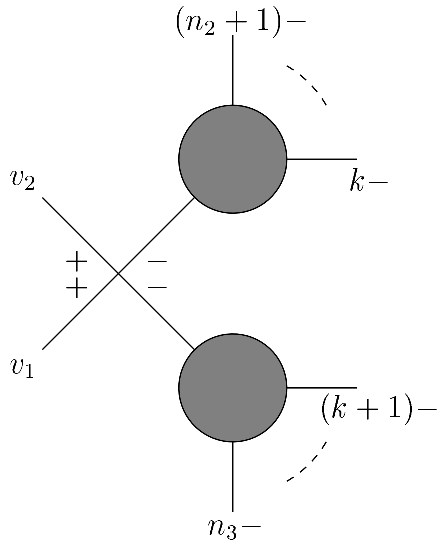

Figure 9: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_1, V_2\rangle$. There is a summation over k.

The next step is to show that all the finite pole terms in $F(\varphi)$ are vanishing. The possible pole terms appear if any factor of $\langle V_1, V_2\rangle\langle V_2, V_3\rangle\langle V_3, V_4\rangle\langle V_4, V_1\rangle$ is vanishing.

Let us consider first the vanishing of $\langle V_1, V_2\rangle$. We denote this set of V_1 and V_2 as v_1 and v_2:

$$v_1 = \sum_{i=n_3+1}^{n+n_1} \lambda_i\,, \qquad v_2 = \sum_{i=n_1+1}^{n_2} \lambda_i\,. \tag{A.10}$$

First let us consider the case when $n_1 + 1 \le r \le n_2$.

As one can see from figure 9 that there is contribution from summing over k and fixing $i = n_1$, $j = n_2$ and $l = n_3$.[6] The residues (ignoring an overall factor $(\varphi_{n_1} - \varphi_{n_1+1})(\varphi_{n_2} - \varphi_{n_2+1})(\varphi_{n_3} - \varphi_{n_3+1})$) for this pole terms are

$$C_1 = \sum_{k=n_2+1}^{n_3-1} (\varphi_k - \varphi_{k+1}) \frac{\langle V_3, V_4\rangle^3}{\langle v_2, V_3\rangle\langle V_4, v_1\rangle}\,. \tag{A.11}$$

Because $\langle v_1, v_2\rangle = 0$, v_1 and v_2 are linearly dependent, we can assume that $v_i = \alpha_i v_0, i = 1, 2$, for some α_i and v_0.

Using this result and $v_1 + v_2 + V_3 + V_4 = 0$, we have

$$C_1 = \frac{(\alpha_1+\alpha_2)^3}{\alpha_1\alpha_2} \sum_{k=n_2+1}^{n_3-1} (\varphi_k - \varphi_{k+1})\langle v_0, V_3\rangle\,. \tag{A.12}$$

By using a little algebra, we can write the summation in eq. (A.12) as following

$$C_1 = \frac{(\alpha_1+\alpha_2)^3}{\alpha_1\alpha_2} \left(\sum_{k=n_2+1}^{n_3-1} \varphi_k \langle v_0, \lambda_k\rangle - \varphi_{n_3}\langle v_0, \sum_{m=n_2+1}^{n_3-1} \lambda_m\rangle \right)\,. \tag{A.13}$$

[6] We note that V_3 and V_4 depend on k: $V_3 = \sum_{i=n_2+1}^{k} \lambda_i$ and $V_4 = \sum_{i=k+1}^{n_3} \lambda_i$.

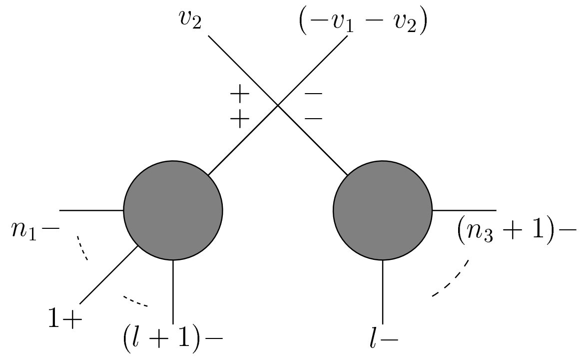

Figure 10: The pole terms $\langle v_1, v_2 \rangle$ from the factor $\langle V_2, V_3 \rangle$. There is a summation over l.

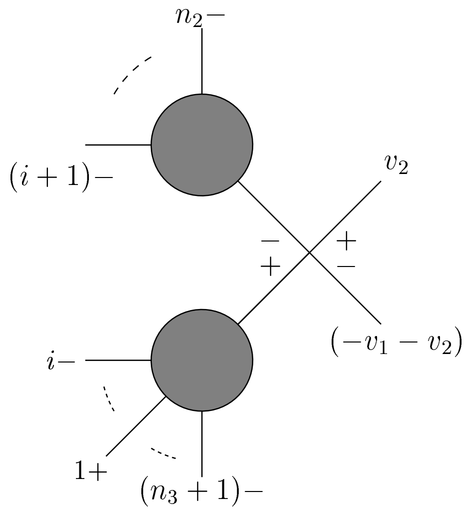

Figure 11: The pole terms $\langle v_1, v_2 \rangle$ from the factor $\langle V_3, V_4 \rangle$. There is a summation over i.

Because $\langle v_0, \sum_{m=n_2+1}^{n_3} \lambda_m \rangle = \langle v_0, V_2 \rangle = 0$, we have $\langle v_0, \sum_{m=n_2+1}^{n_3-1} \lambda_m \rangle = -\langle v_0, \lambda_{n_3} \rangle$. Then

$$C_1 = \frac{(\alpha_1 + \alpha_2)^3}{\alpha_1 \alpha_2} \langle v_0, \sum_{k=n_2+1}^{n_3} \varphi_k \lambda_k \rangle \,. \tag{A.14}$$

Unlike the special case discussed in [12], here for $n_1 + 1 \leq r \leq n_2$, there are pole terms from the vanishing of the factor $\langle V_2, V_3 \rangle$ in figure 10 by setting $V_2 = v_2$ and $V_3 = -v_1 - v_2$ and summing over l. This gives the following contribution:

$$\begin{aligned} C_2 &= \sum_{l=n_3+1}^{n} (\varphi_l - \varphi_{l+1}) \frac{\langle -v_1 - v_2, V_4 \rangle^3}{\langle v_1 - V_4, v_2 \rangle \langle V_4, v_1 - V_4 \rangle} \\ &= \frac{(\alpha_1 + \alpha_2)^3}{\alpha_1 \alpha_2} \sum_{l=n_3+1}^{n} (\varphi_l - \varphi_{l+1}) \langle v_0, V_4 \rangle \\ &= \frac{(\alpha_1 + \alpha_2)^3}{\alpha_1 \alpha_2} \left\langle v_0, \sum_{l=n_3+1}^{n} \varphi_l \lambda_l - \varphi_1 \sum_{m=n_3+1}^{n} \lambda_m \right\rangle . \end{aligned} \tag{A.15}$$

The third piece of the pole terms is from the vanishing of $\langle V_3, V_4 \rangle$ in figure 11 by setting $V_3 = v_2$ and $V_4 = -v_1 - v_2$ and summing over i. This give the following contribution:

$$C_3 = \frac{(\alpha_1 + \alpha_2)^3}{\alpha_1 \alpha_2} \left\langle v_0, \varphi_1 \sum_{m=n_3+1}^{n+1} \lambda_m + \sum_{i=2}^{n_1} \varphi_i \lambda_i \right\rangle . \tag{A.16}$$

The last piece of the pole terms is from the vanishing of $\langle V_4, V_1 \rangle$ in figure 12 and figure 13 by setting $V_1 = v_1$ and $V_4 = -v_1 - v_2$ and summing over j. The contribution corresponds to the figure 12 is

$$C_4 = \frac{(\alpha_1 + \alpha_2)^3}{\alpha_1 \alpha_2} \left\langle v_0, \sum_{j=n_1+1}^{r-1} \varphi_j \lambda_j - \varphi_r \sum_{m=n_1+1}^{r-1} \lambda_m \right\rangle , \tag{A.17}$$

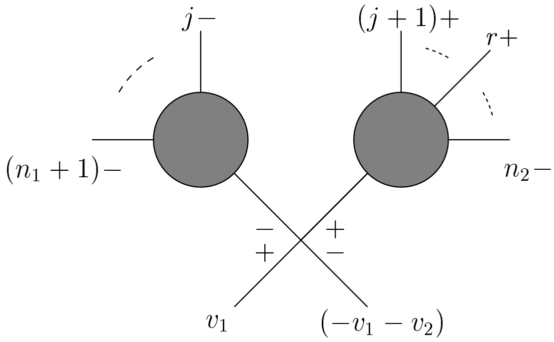

Figure 12: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_4, V_1\rangle$. There is a summation over j from n_1+1 to $r-1$.

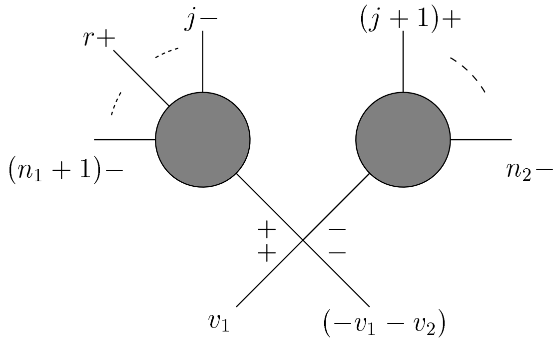

Figure 13: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_4, V_1\rangle$. There is a summation over j from r to n_2-1.

and the contribution corresponds to the figure 13 is

$$C_5 = \frac{(\alpha_1+\alpha_2)^3}{\alpha_1\alpha_2}\left\langle v_0, \varphi_r \sum_{m=n_1+1}^{r} \lambda_j + \sum_{j=r+1}^{n_2} \varphi_j\lambda_j \right\rangle . \tag{A.18}$$

By summing all these five contributions together, we have

$$C_1+C_2+C_3+C_4+C_5 = \frac{(\alpha_1+\alpha_2)^3}{\alpha_1\alpha_2}\left\langle v_0, \sum_{i=1}^{n} \varphi_i\lambda_i \right\rangle = 0 . \tag{A.19}$$

This prove that there is no pole terms for the function $F(\varphi)$ when $n_1+1 \le r \le n_2$. The case for $n_2+1 \le r \le n_3$ is similar. The case for $r = n$ or $r = 2$ is also similar and the algebra becomes a little bit easier because we need only consider three different kinds of diagram in these two cases as shown in [12]. When $3 \le r \le n_1$, the situation is even easier because in this case all of the possible contribution to the pole terms is come from the diagrams in figure 14, but actually the numerator has a zero of order 4 while the denominator has a zero of order 1. So there is no singularity in every individual term. Similarly there is no pole terms when $n_3+1 \le r \le n-1$. In summary we proved that all the finite pole terms in $F(\varphi)$ are vanishing in all different cases.

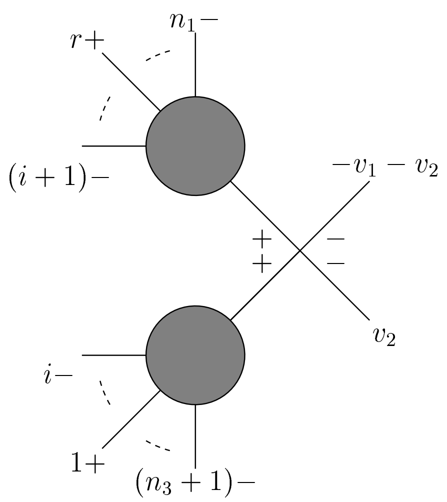

Figure 14: All of the possible pole terms are come from this kind of diagrams when $3 \le r \le n_1$, but actually there is no singularity.

So $F(\varphi)$ is independent of φ_j for $2 \le j \le n$, $j \ne r$. Now we compute $F(\varphi)$ explicitly by choosing a special set of φ_j for $2 \le j \le n$, $j \ne r$.[7] A convenient choice is as follows:

$$\varphi_2 = \cdots = \varphi_{r-1} = x\,, \qquad \varphi_{r+1} = \cdots = \varphi_n = y\,. \tag{A.20}$$

[7]The case when $r = n$ or $r = 2$ is a little bit different, and it was done in [12].

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By using eq. (A.8) and eq. (A.9), we have

$$\lambda_1 = \frac{(x-\varphi_r)\lambda + (y-\varphi_r)\mu}{\varphi_r - \varphi_1}\,, \qquad \lambda_r = -\frac{(x-\varphi_1)\lambda + (y-\varphi_1)\mu}{\varphi_r - \varphi_1}\,, \tag{A.21}$$

by setting $\sum_{i=2}^{r-1} \lambda_i = \lambda$ and $\sum_{j=r+1}^{n} \lambda_j = \mu$. In the following we will use the new variables x' and y' which we define as

$$x' = \frac{x-\varphi_r}{\varphi_r - \varphi_1}\,, \qquad y' = \frac{y-\varphi_r}{\varphi_r - \varphi_1}\,, \tag{A.22}$$

then we have

$$\frac{x-\varphi_1}{\varphi_r - \varphi_1} = x' + 1\,, \qquad \frac{y-\varphi_1}{\varphi_r - \varphi_1} = y' + 1\,. \tag{A.23}$$

Now we show that for generic x and y (and generic λ_i which we don't say explicitly) all the possible $\langle V_1, V_2\rangle$, $\langle V_2, V_3\rangle$, $\langle V_3, V_4\rangle$ and $\langle V_4, V_1\rangle$ for different choices of i, j, k and l are non-vanishing.

In order to do this, let's define a matrix $\{t_{ij}\}_{(1\le i\le n, 1\le j\le 4)}$. If λ_i is one of the terms in V_j then we define $t_{ij} = 1$. Otherwise we set $t_{ij} = 0$. Then

$$\begin{aligned} V_j = \sum_{i=1}^{n} t_{ij}\lambda_i = \sum_{i=2}^{r-1} \lambda_i [t_{ij} + t_{1j}x' - t_{rj}(x'+1)] + \\ + \sum_{i=r+1}^{n} \lambda_i [t_{ij} + t_{1j}y' - t_{rj}(y'+1)]\,. \end{aligned} \tag{A.24}$$

So for generic x, y and λ_i, V_i and V_{i+1} are linearly independent and so $\langle V_i, V_{i+1}\rangle$ is non-vanishing.

For our choice of φ, the possible non-vanishing factor for $(\varphi_i - \varphi_{i+1})(\varphi_j - \varphi_{j+1})(\varphi_k - \varphi_{k+1})$ is from $i = 1$, $j = r-1$, $k = r$ and $l = n$ only. Then we have

$$F(\varphi) = (\varphi_1 - x)(x - \varphi_r)(\varphi_r - y)(y - \varphi_1)\frac{\langle \lambda, \mu\rangle}{\langle \lambda_1, \lambda\rangle\langle \lambda, \lambda_r\rangle\langle \lambda_r, \mu\rangle\langle \mu, \lambda_1\rangle}\,. \tag{A.25}$$

By using eq. (A.21), we have

$$\langle \lambda_1, \lambda\rangle = -\frac{y-\varphi_r}{\varphi_r - \varphi_1}\langle \lambda, \mu\rangle\,, \tag{A.26}$$

$$\langle \lambda, \lambda_r\rangle = -\frac{y-\varphi_1}{\varphi_r - \varphi_1}\langle \lambda, \mu\rangle\,, \tag{A.27}$$

$$\langle \lambda_r, \mu\rangle = -\frac{x-\varphi_1}{\varphi_r - \varphi_1}\langle \lambda, \mu\rangle\,, \tag{A.28}$$

$$\langle \mu, \lambda_1\rangle = -\frac{x-\varphi_r}{\varphi_r - \varphi_1}\langle \lambda, \mu\rangle\,, \tag{A.29}$$

and so we have

$$F(\varphi) = (\varphi_r - \varphi_1)^4\,. \tag{A.30}$$

This completes the proof of eq. (2.15).

B. The computation of $P(\Gamma)$

In this appendix, we will give the proof of eq. (5.3). We assume that the off-shell graviton has positive helicity. The case when it has negative helicity is similar.

When $n = 3$, we have $\lambda_1 + \lambda_2\phi_2 + \lambda_3\phi_3 = 0$, so $\langle 1, 2\rangle = \phi_3\langle 2, 3\rangle$, $\langle 3, 1\rangle = \phi_2\langle 2, 3\rangle$. Then

$$A_3(1+, 2-, 3-) = \frac{1}{\phi_2^2\phi_3^2}\langle 2, 3\rangle^2 = \frac{p_1^2}{\phi_2^2\phi_3^2}\frac{\langle 2, 3\rangle}{[2, 3]}\,. \tag{B.1}$$

There is only one inequivalent undirected tree with vertex set $\{2, 3\}$. There is only one edge e_{23} in this tree. It is easy to see that

$$P(\Gamma) = p(2)p(3)p(e_{23}) = \phi_2^{-4}\phi_3^{-4}\left(\phi_2^2\phi_3^2\frac{\langle 2, 3\rangle}{[2, 3]}\right) = \frac{1}{\phi_2^2\phi_3^2}\frac{\langle 2, 3\rangle}{[2, 3]}\,. \tag{B.2}$$

So $M_3(1+, 2-, 3-) = p_1^2 P(\Gamma)$. This completes the proof for eq. (5.3) when $n = 3$. Assuming that it is true for all $k \leq n$, we will prove that it is also true for $k = n + 1$.

We use the diagram decomposition in figure 15 to calculate A_{n+1}. So we have

$$\begin{aligned} A_{n+1} = & \sum_{\{i_1,\ldots,i_m\}} A_{m+1}(-q_1, i_1, \ldots, i_m)\,\frac{1}{q_1^2}\times \\ & \times A_{n-m+1}(-q_2, j_1, \ldots, j_{n-m})\frac{1}{q_2^2}V_3(1, q_1, q_2)\,, \end{aligned} \tag{B.3}$$

where $i_1, \ldots, i_m$ are any m $(1 \leq m \leq n-1)$ gravitons in $\{2, \ldots, n+1\}$, $j_1, \ldots, j_{n-m}$ are the remaining $n - m$ ones. q_1 and q_2 are two internal lines with momenta $q_1 = \sum_{k=1}^{m} p_{i_k}$ and $q_1 = \sum_{k=1}^{n-m} p_{j_k}$ respectively. We denote $\{i_1, \ldots, i_m\}$ as S_1 and $\{j_1, \ldots, j_{n-m}\}$ as S_2. It is easy to see that when $m = 1$ $(m = n-1)$, if we understand $A_2(-q_1, i_1)$ $(A_2(-q_2, j_1))$ as $1/\phi_{i_1}^4$ $(1/\phi_{j_1}^4)$, then these two degenerate cases are included correctly.

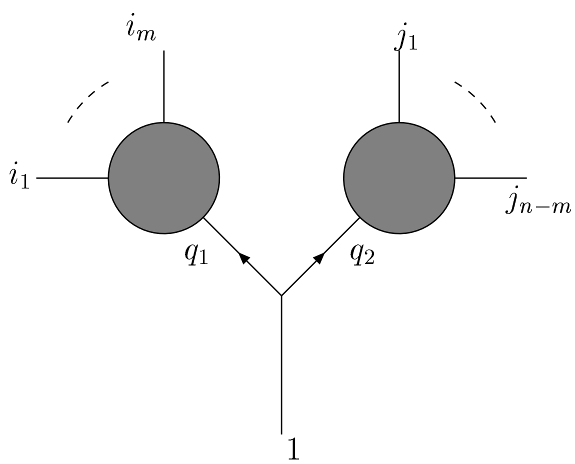

Figure 15: The decomposition for the amplitude for gravitons using in the computation of $P(\Gamma)$.

By using the assumed result for all less multi-graviton amplitudes, we can get

$$A_{n+1} = \sum_{\Gamma_1,\Gamma_2} P(\Gamma_1)P(\Gamma_2)V_3(1,q_1,q_2)\,, \tag{B.4}$$

where Γ_1 and Γ_2 are two disjoint undirected trees subject to the constraint that the union of their vertex set are $\{2,3,\ldots,n\}$. (In this appendix, the summation over Γ_1 and Γ_2 is always understood to be under this constraint.) When $m=1$ ($m=n-1$) the tree Γ_1 (Γ_2) includes only one vertex and the there are no edges in Γ_1 (Γ_2). These degenerate cases are included correctly, too.

From eq. (5.2), we have

$$\begin{aligned} V_3(1,q_1,q_2) &= \langle\lambda_{q_1},\lambda_{q_2}\rangle^2 = \left(\sum_{i\in V_1,j\in V_2}\langle\lambda_{p_i},\lambda_{p_j}\rangle\right)^2 \\ &= \sum_{i,j\in V_1,k,l\in V_2}\langle\lambda_{p_i},\lambda_{p_k}\rangle\langle\lambda_{p_j},\lambda_{p_l}\rangle\,, \end{aligned} \tag{B.5}$$

where V_i is the vertex set $V(\Gamma_i)$ of graph Γ_i ($i=1,2$). So we have

$$A_{n+1} = \sum_{\Gamma_1,\Gamma_2} P(\Gamma_1)P(\Gamma_2)\sum_{i,j\in V_1,k,l\in V_2}\langle\lambda_{p_i},\lambda_{p_k}\rangle\langle\lambda_{p_j},\lambda_{p_l}\rangle\,. \tag{B.6}$$

By denoting $2p_i\cdot p_j = \langle i,j\rangle[i,j]$ as s_{ij} and then using

$$p_1^2 = \left(\sum_{i=2}^{n+1} p_i\right)^2 = \sum_{2\le i<j\le n+1} s_{ij}\,, \tag{B.7}$$

we get

$$p_1^2 \sum_{\Gamma,V(\Gamma)=S(n+1)} P(\Gamma) = \sum_{\Gamma,V(\Gamma)=S(n+1)} P(\Gamma)\sum_{i,j\in V(\Gamma),i<j} s_{ij}\,. \tag{B.8}$$

For any $i,j\in V(\Gamma)$, there is an unique path in the tree Γ jointing i and j. We denote this path as $v_1=i,v_2,\ldots,v_{k-1},v_k=j$. By using the following identity:

$$\frac{[i,j]}{\phi_i\phi_j} = \frac{[i,k]}{\phi_i\phi_k} + \frac{[k,j]}{\phi_k\phi_j}\,, \tag{B.9}$$

we get

$$\frac{[i,j]}{\phi_i\phi_j} = \sum_{s=1}^{k-1}\frac{[v_s,v_{s+1}]}{\phi_{v_s}\phi_{v_{s+1}}}\,. \tag{B.10}$$

So

$$\begin{aligned} P(\Gamma)s_{ij} &= P(\Gamma)\langle i,j\rangle\phi_i\phi_j\frac{[i,j]}{\phi_i\phi_j} \\ &= \sum_{s=1}^{k-1} P(\Gamma)\langle i,j\rangle\phi_i\phi_j\frac{[v_s,v_{s+1}]}{\phi_{v_s}\phi_{v_{s+1}}}\,. \end{aligned} \tag{B.11}$$

If we move away an edge from a tree, we can get two disjoint trees. Conversely, assuming that Γ_1 and Γ_2 are two disjoint trees, if we connect one vertex in Γ_1 with another vertex in Γ_2 by an edge, we get a bigger tree. We denote the two trees obtained by moving away the edge $e_{v_s v_{s+1}}$ as $\Gamma_1(v_s)$ and $\Gamma_2(v_s)$. Then we have

$$\begin{aligned} P(\Gamma)s_{ij} &= \sum_{s=1}^{k-1} P(\Gamma_1(v_s))P(\Gamma_2(v_s))\frac{\langle v_s, v_{s+1}\rangle}{[v_s, v_{s+1}]}\phi_{v_s}^2\phi_{v_{s+1}}^2\langle i,j\rangle\phi_i\phi_j\frac{[v_s, v_{s+1}]}{\phi_{v_s}\phi_{v_{s+1}}} \\ &= \sum_{s=1}^{k-1} P(\Gamma_1(v_s))P(\Gamma_2(v_s))\langle v_s, v_{s+1}\rangle\phi_{v_s}\phi_{v_{s+1}}\langle i,j\rangle\phi_i\phi_j\,. \end{aligned} \tag{B.12}$$

From this we can get

$$\begin{aligned} \sum_{\Gamma, V(\Gamma)=S(n+1)} P(\Gamma)p_1^2 &= \sum_{\Gamma, V(\Gamma)=S(n+1)} P(\Gamma) \sum_{i,j\in V(\Gamma), i<j} s_{ij} \\ &= \sum_{\Gamma_1,\Gamma_2} P(\Gamma_1)P(\Gamma_2) \sum_{i,j\in V_1, k,l\in V_2} \langle\lambda_{p_i},\lambda_{p_k}\rangle\langle\lambda_{p_j},\lambda_{p_l}\rangle \\ &= A_{n+1}\,, \end{aligned} \tag{B.13}$$

as announced. This completes the proof of eq. (5.3) by mathematical induction.

References

[1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, hep-th/0312171.

[2] R. Penrose, *Twistor algebra*, *J. Math. Phys.* **8** (1967) 345; *The central programme of twistor theory*, *Chaos, Solitons, and Fractal* **10** (1999) 581.

[3] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, hep-th/0403047.

[4] S.J. Parke and T.R. Taylor, *An amplitude for N gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[5] F.A. Berends and W.T. Giele, *Recursive calculations for processes with N gluons*, *Nucl. Phys.* **B 306** (1988) 759.

[6] M.L. Mangano and S. Parke, *Multiparton amplitudes in guage theories*, *Phys. Rept.* **200** (1991) 30.

[7] Z. Bern, *String based perturbative methods for gauge theories*, hep-ph/9304249.

[8] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[9] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109 [hep-ph/9602280].

[10] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040]; Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop helicity amplitudes for quark gluon scattering in QCD and gluino gluon scattering in supersymmetric yang- mills theory*, *J. High Energy Phys.* **06** (2003) 028 [hep-ph/0304168].

[11] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.

[12] C.-J. Zhu, *The googly amplitudes in gauge theory*, *J. High Energy Phys.* **04** (2004) 032 [hep-th/0403115].

[13] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the b-model in twistor space*, *J. High Energy Phys.* **04** (2004) 012 [hep-th/0402016];
R. Roiban and A. Volovich, *All googly amplitudes from the b-model in twistor space*, hep-th/0402121.

[14] R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, *Phys. Rev.* **D 70** (2004) 026009 [hep-th/0403190].

[15] N. Berkovits, *An alternative string theory in twistor space for $N = 4$ super- Yang-Mills*, *Phys. Rev. Lett.* **93** (2004) 011601 [hep-th/0402045].

[16] N. Berkovits and L. Motl, *Cubic twistorial string field theory*, *J. High Energy Phys.* **04** (2004) 056 [hep-th/0403187].

[17] M. Aganagic and C. Vafa, *Mirror symmetry and supermanifolds*, hep-th/0403192.

[18] E. Witten, *Parity invariance for strings in twistor space*, hep-th/0403199.

[19] A. Neitzke and C. Vafa, *$N = 2$ strings and the twistorial Calabi-Yau*, hep-th/0402128.

[20] N. Nekrasov, H. Ooguri and C. Vafa, *S-duality and topological strings*, hep-th/0403167.

[21] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *J. High Energy Phys.* **05** (2004) 070 [hep-th/0404072].

[22] S. Gukov, L. Motl and A. Neitzke, *Equivalence of twistor prescriptions for super Yang-Mills*, hep-th/0404085.

[23] W. Siegel, *Untwisting the twistor superstring*, hep-th/0404255.

[24] S. Giombi, R. Ricci, D. Robles-Llana and D. Trancanelli, *A note on twistor gravity amplitudes*, hep-th/0405086.

[25] A.D. Popov and C. Saemann, *On supertwistors, the Penrose-Ward transform and $N = 4$ super Yang-Mills theory*, hep-th/0405123.

[26] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, hep-th/0406051.

[27] I. Bena, Z. Bern and D.A. Kosower, *Twistor-space recursive formulation of gauge theory amplitudes*, hep-th/0406133.

[28] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, hep-th/0406175.

[29] F. Cachazo, P. Svrcek and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, hep-th/0406177.

[30] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, *Supergravity and the S-matrix*, *Phys. Rev.* **D 15** (1977) 996.

[31] M.T. Grisaru and H.N. Pendleton, *Some properties of scattering amplitudes in supersymmetric theories*, *Nucl. Phys.* **B 214** (1977) 81.

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[32] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, hep-th/0407027.

[33] J.-B. Wu and C.-J. Zhu, *MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos*, hep-th/0406146.

[34] F.A. Berends, W.T. Giele and H. Kuijf, *On relations between multi-gluon and multigraviton scattering*, *Phys. Lett.* **B 211** (1988) 91.

[35] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A relation between tree amplitudes of closed and open strings*, *Nucl. Phys.* **B 269** (1986) 1.

[36] M.L. Mangano, S.J. Parke and Z. Xu, *Duality and multi-gluon scattering*, *Nucl. Phys.* **B 298** (1988) 653.